Sketch of solution to Homework 1

Q1 Suppose f is one-to-one. Define $g : Y \to X$ by $g(y) = x$ if $y = f(x)$ for some $x \in X$, otherwise $g(y) = x_0$ for some x_0 fixed. Since f is one-to-one, g is well-defined. Obviously, $g \circ f = id$.

Suppose there is $g : Y \to X$ such that $g \circ f = id$. Let $x_0, x_1 \in X$ be such that $f(x_0) = f(x_1)$. Then $x_0 = g \circ f(x_0) = g \circ f(x_1) = x_1$.

Q4 Let $x \in B \cap \left[\bigcup_{A \in C} A\right]$, then $x \in B$ and $x \in \bigcup_{A \in C} A$. In particular, $x \in A'$ for some $A' \in C$. Hence,

$$x \in A' \cap B \subset \bigcup_{A \in C} (B \cap A).$$

The opposite direction is similar.

Q7 (a) Let $x \in f^{-1}[\cup B_\lambda]$. Then $f(x) \in \cup B_\lambda$ or equivalently there is $B_{\lambda'}$ such that

$$f(x) \in B_{\lambda'} \subset \cup B_\lambda.$$

Let $x \in \cup f^{-1}(B_\lambda)$, then $x \in f^{-1}(B_{\lambda'})$ for some λ'. Therefore,

$$f(x) \in B_{\lambda'} \subset \cup B_\lambda.$$

(b) Let $x \in f^{-1}(\cap B_\lambda)$, then $f(x) \in \cap B_\lambda$. Equivalently, $f(x) \in B_\lambda$ for all λ meaning that

$$x \in f^{-1}(B_\lambda)$$

for all λ. Hence, $x \in \cap f^{-1}(B_\lambda)$.

Let $x \in \cap f^{-1}(B_\lambda)$, then $x \in f^{-1}(B_\lambda)$ for all λ. Hence $f(x) \in \cap B_\lambda$, $x \in f^{-1}(\cap B_\lambda)$.

(c)

$$\begin{aligned} & x \in f^{-1}(B^c) \\ \iff & f(x) \in B^c \\ \iff & f(x) \notin B. \end{aligned}$$

If $x \in f^{-1}(B)$, then $f(x) \in B$ which is impossible. The opposite direction is similar.

Q8 (a) Let $x \in f(f^{-1}(B))$, then $x = f(y)$ for some $y \in f^{-1}(B)$. Therefore, $f(y) \in B$. Hence $x \in B$. Similarly, let $x \in A$, then $f(x) \in f(A)$. Hence $x \in f^{-1}(f(A))$.

(b) Take $f : \{1\} \to \{1, 2\}$ by $f(1) = 1$. We see that the first inequality is strict.

To see that the second inequality can be strict. We can simply take some function which is not injective.

(c) Let $y \in B$, by assumption there is $x \in X$ such that $f(x) = y$. That is $x \in f^{-1}(B)$. Then $y = f(x) \in f(f^{-1}(B))$.